Invariant Differential Systems and Canonical Forms of E. Cartan

by

H. H. Johnson

Technical Report No. 19
April 18, 1963

Contract Nonr 477(15)
Project Number NR 043 186

Department of Mathematics
University of Washington
Seattle 5, Washington

Invariant Differential Systems and Canonical Forms of E. Cartan

by

H. H. Johnson

One of the early applications of continuous transformation groups and pseudo groups was to invariant systems of ordinary and partial differential equations. Much of this work was done by S. Lie [5, 6] and E. Vessiot [7] before Cartan's contributions to infinite continuous pseudo groups [1]. The purpose of this paper is to develop parts of the older theory of Lie and Vessiot in Cartan's context. We show the essential role played by Cartan's canonical forms in determining invariant systems. Automorphic systems can be completely described in a manner similar to Lie's, but Cartan's involutiveness together with an additional hypothesis yield more complete results than in the older theory. Also, following Cartan the theory takes a "coordinate-free" form. Definitions will be those of M. Kuranishi [3, 4]. In this paper we will always assume manifolds, functions, and forms are real and infinitely differentiable.

Invariant Systems

M is a manifold. G is a continuous infinite pseudo group of local transormations of M into itself which is complete, i.e. restrictions of transformations to open subsets are also considered to be in G. [4] Σ is an ideal of exterior differential forms on M which is closed under exterior differentiation and generated by 1-forms $\phi^1,\dots,\phi^m$ and their exterior derivatives [3]. A <u>solution</u>

of Σ is a submanifold $F: N \to M$ of M such that $F^*\phi = 0$ for every ϕ in Σ. The only additional condition imposed on Σ is that if ϕ is a 1-form which is zero on every solution of Σ, then ϕ is a linear combination of $\phi^1,\ldots,\phi^m$. (Real analytic systems studied by E. Cartan have this property.) One can show that if Σ satisfies this property on M, then it continues to do so when restricted to an open subset of M.

Definition 1. Σ is invariant under G if for every solution $F: N \to M$ and every f in G having domain U which intersects the range of F, $fF: F^{-1}(U) \to M$ is also a solution of Σ.

Definition 2. It is assumed that G is in normal form. This means that there are on M real-valued functions $y^1,\ldots,y^p$ and 1-forms $\omega^1,\ldots,\omega^q$, $\pi^1,\ldots,\pi^r$ such that at every point p in M, $dy^1,\ldots,dy^p$, $\omega^1,\ldots,\omega^q$, $\pi^1,\ldots,\pi^r$ form a basis of the cotangent space and G satisfies:

(1) f is in G if and only if $y^i f = y^i$, $f^*\omega^j = \omega^j$ for every $i = 1,\ldots,p$, $j = 1,\ldots,q$;

(2) $d\omega^i = c^i{}_{jk}\,\omega^j \wedge \omega^k - a^i{}_{j\rho}\,\omega^j \wedge \pi^\rho$,
where the $c^i{}_{jk}$ anf $a^i{}_{j\rho}$ are functions of $y^1,\ldots,y^p$;

(3) $a^i{}_{j\rho}$ form an involutive system [1]. That is, on $M \times M$ with P_1 and P_2 the projections of $M \times M$ onto the first and second factors, respectively, the system Ω of exterior differential forms generated by $P_1^*\omega^i - P_2^*\omega^i$, $y^j P_1 - y^j P_2$, all i, j, with $P_1(M)$ as independent variables, is in involution.

For our purposes, and because we are in the C^{∞} - category, we impose the additional hypothesis:

(4) every integral $\Phi : N \to M \times M$ of Ω such that $P_1\Phi : N \to M$ is a submanifold of M is contained in a (maximal) integral of Ω having $P_1(M)$ as independent variables. That is, there exists a solution $F: M \to M \times M$ with P_1F = identity, and a map $T: N \to M$ so that $\Phi = FT$.

<u>Remark</u>. The assumption that G is in normal form does not greatly restrict generality. Many pseudo groups on M may be prolonged to a manifold $M \times M'$. If $P: M \times M' \to M$ is the projection, and if Σ on M is invariant under G, then $P^*(\Sigma)$ is invariant under the prolonged pseudo group. All of the pseudo groups studied by Cartan could be prolonged to normal form. Also, using his normal prolongations one may as well assume that $\phi^1,\dots,\phi^m$ can be expressed as a linear combination of $dy^1,\dots, dy^p, \omega^1,\dots, \omega^q$.

<u>Theorem</u> 1. <u>If</u>

$$\phi^i = s^i{}_k dy^k - t^i{}_j \omega^j, \ i = 1,\dots,m,$$

<u>where</u> $s^i{}_k$ <u>and</u> $t^i{}_j$ <u>are functions of</u> $y^1,\dots,y^p$, <u>then the ideal</u> Σ <u>generated by</u> $\phi^1,\dots,\phi^m$ <u>is invariant</u>. <u>Conversely</u>, <u>every invariant ideal generated by</u> 1-<u>forms which are linear combinations of</u> $dy^1,\dots,dy^p, \omega^1,\dots, \omega^q$, <u>is generated by</u> 1-<u>forms of the type above</u>.

<u>Proof</u>. The first assertion follows from condition (1) in the definition of normal form.

Conversely, suppose ϕ is a 1-form in an invariant ideal Σ and f is in G having domain U. Then $f^*\phi$ is zero on every solution of Σ restricted to U. Hence $f^*\phi$ is a linear combination of the generators of Σ.

Let $\beta^1,\dots,\beta^m$ denote the generators of Σ, and let $\psi^1,\dots,\psi^{p-q}$ denote $dy^1,\dots,dy^p$, $\omega^1,\dots,\omega^q$ in some order. Given that the β^i are linear combinations of the ψ^j, it can be supposed that

$$\beta^i = \psi^i - \sum_{k=1}^{p+q-m} A^i{}_k\, \psi^{m+k},\ i = 1,\dots,m.$$

If f is in G, then

$$f^*\beta^i = \psi^i - \sum_{k=1}^{p+q-m} (A^i{}_k f)\psi^{m-k},\ i = 1,\dots,m.$$

But also $f^*\beta^i$ is a linear combination of $\beta^1,\dots,\beta^m$:

$$f^*\beta^i = c^i{}_h\beta^h = \sum_{h=1}^{m} c^i{}_h\, \psi^h - \sum_{h=1}^{m}\sum_{k=1}^{p+q-m} c^i{}_h A^h{}_k \psi^{m-k},$$

$i = 1,\dots,m$. Since $dy^1,\dots,dy^p$, $\omega^1,\dots,\omega^q$ are linearly independent at every point of M, $c^i{}_h = \delta^i{}_h$, and hence $f^*\beta^i = \beta^i$ for every f in G. Moreover, $A^i{}_k f = A^i{}_k$, so these coefficients are invariants and consequently functions of $y^1,\dots,y^p$. Q.E.D.

Automorphic Systems

<u>Definition</u> 3. An <u>exterior differential system on</u> $M \times N$ <u>with independent variables</u> N is a pair (Σ', N) where Σ' is an ideal of exterior differential forms on $M \times N$ closed under exterior differentiation, and N is a manifold. A <u>solution</u> of (Σ', N) is a submanifold $F: V \to M \times N$ where V is an open subset of N, $P_N F$ = identity, $P_M F$ is a submanifold of M, and $F^*\phi = 0$ for every ϕ in Σ'. (P_N and P_M denote the projections of MN onto N and M, respectively.)

<u>Definition</u> 4. Two solutions $F_1: V \to M \times N$ and $F_2: V \to M \times N$ of (Σ', N) are called <u>equivalent under</u> G', a pseudo group on $M \times N$, if for every p in V there is an f' in G' whose domain U contains $F_1(p)$ such that $f'F_1 = F_2$ on $F_1^{-1}(U)$.

<u>Definition</u> 5. (Σ', N) is <u>automorphic under</u> G' if every two solutions $F_1: V \to MN$ and $F_2: V \to MN$ are equivalent, and if Σ' is invariant under G'.

Now let G be as before a pseudo group on M. For every f in G having domain U, consider on $U \times N$ the transformation $f'(u,n) = (f(u),n)$, u in U, n in N. That is, $P_M f' = f P_M$. The collection of all such f' together with their restrictions to open subsets forms a complete pseudo group G' called the <u>trivial</u> isomorphic prolongation of G to $M \times N$.

<u>Proposition</u> 1. Let G on M be in normal form with invariants $y^1, \cdots, y^p$, $\omega^1, \cdots, \omega^q$. Let $F_Q: N \to M$ be any submanifold of M. Let G' denote the trivial prolongation of G to $M \times N$. Define

Σ' to be the ideal of exterior differential forms, closed under exterior differentiation, on $M \times N$ generated by

$$y^i P_M - y^i F_o P_N, \; i = 1,\cdots,p,$$

$$P_M^* \omega^j - P_N^* F_o^* \omega^j, \; j = 1,\cdots,q.$$

Then (Σ', N) is automorphic under the trivial prolongation G'.

<u>Proof</u>. Let $F_1: V \to M$ and $F_2: V \to M$ be two solutions of (Σ', N). Then $y^i P_M F_1 = y^i F_o P_N F_1 = y^i F_o$ and $y^i P_M F_1 = y^i F_o$. Hence $y^i P_M F_1 = y^i P_M F_2$, $i = 1,\cdots,p$. Similarly, $(P_M F_1)^* \omega^j = (P_M F_2)^* \omega^j$.

Now consider the ideal Ω on $M \times M$ of Definition 2. The submanifold $\Phi: V \to M \times M$ defined by $\Phi(q) = (P_M F_1(q), P_M F_2(q))$ is an integral submanifold of Ω by the remarks of the previous paragraph. Moreover, $P_1 \Phi = P_M F_1$ is a submanifold of Σ, and hence by (4) in Definition 2, Φ is contained in a solution of maximal dimensions. This solution defines an element f of G, and $f P_M F_1 = P_M F_2$. It follows that $f' F_1 = F_2$. The fact that Σ' is invariant is a consequence of Theorem 1. Q.E.D.

<u>Proposition</u> 2. Let Σ be invariant on M under G and have a solution $F_o: N \to M$. Let Σ' be the ideal constructed in Proposition 1 from F_o. Let Σ'' denote the ideal $P_M^*(\Sigma)$ on $M \times N$. If (Σ'', N) is automorphic under G', then (Σ', N) and (Σ'', N) have the same solutions.

<u>Proof</u>. If $F: V \to M \times N$ is a solution of (Σ'', N), it must be equivalent to F_o' defined by $F_o'(q) = (F_o(q), q)$. Hence locally one can write $F = f' F_o'$ for some f' in G'. F_o' is trivially a

solution of (Σ',N), and Σ' is invariant under G'. Hence $f'F_0 = F$ is locally a solution of (Σ',N). But then it must be a global solution. The same reasoning shows any solution of (Σ',N) to be a solution of (Σ'',N). Q.E.D.

These two propositions provide a complete description of automorphic systems. The author has been unable to find a similar extension of the Lie-Vessiot theory of decomposition of invariant systems by quotient pseudo groups into resolvants and automorphic systems. [2]